Configuration graphs and random forests

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July 6, 2022

Configuration graphs are widely used to simulate complex communication networks such as the Internet. Methods from the theory of branching processes have been successfully applied to study the structure and dynamics of configuration graphs. We consider a homogeneous critical Galton-Watson branching process G, starting with N particles and denote by ξ the random variable equal to the number of offspring distribution of all particles. Suppose that ξ has the distribution

$$p_k = \mathbf{P}\{\xi = k\} = \frac{h(k+1)}{(k+1)^\tau}, \quad k = 0, 1, 2, \ldots, \quad \tau \in (2, 3),$$

where $h(x)$ is a slowly varying at infinity function. Consider the subset of trajectories of G that contain $N + n$ particles. As a result, we obtain a random forest consisting of N rooted trees and containing n non-root vertices. We prove limit theorems for the maximum tree size and for the number of trees with a given size for various relations between N and n as they tend to infinity. To formulate one of our results we will introduce the necessary notation. Let η be the maximum tree size. Denote by $\xi(\lambda)$ a random variable with distribution

$$\mathbf{P}\{\xi(\lambda) = k\} = \frac{\lambda^k p_k}{F(\lambda)}, \quad k = 0, 1, 2, \ldots,$$

where $0 < \lambda < 1$ and

$$F(\lambda) = \sum_{k=0}^{\infty} p_k \lambda^k.$$

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We choose a unique solution of the equation

$$\mathbf{E}\xi(\lambda) = \frac{n}{N+n}$$

for the value of the parameter λ. We denote also $\beta = -\ln(\lambda/F(\lambda))$. Set

$$L = L(N,n,\delta) = \min\left(n(1-\lambda)^{\tau-1+\delta}, \frac{N^2}{n}(1-\lambda)^{3-\tau+\delta}\right),$$

where $\delta > 0$. Assume that $g(x)$ is the density of the stable law with exponent $\tau - 1$ and characteristic function

$$\exp\left\{-\frac{\Gamma(2-\tau)}{\tau-1}|t|^{\tau-1}\left(1 - i\frac{t}{|t|}\tan\frac{\pi(\tau-1)}{2}\right)\cos\frac{\pi(\tau-1)}{2}\right\}.$$

We introduce a sequence $B_r, \quad r = 1, 2, \ldots,$ that tend to infinity and satisfies the condition $B_r \sim (rh(B_r))^{1/(\tau-1)}$ as $r \to \infty$. The following statement hold.

Theorem. *Let $N, n \to \infty$ in such a way that $n/N \to \infty$ and let there exist positive constants δ, ω, $0 < \omega < 1/2$, such that $L \to \infty$, $\beta > N^{-\omega}$. Assume that $r = r(N,n)$ are chosen so that*

$$\frac{Ng(0)}{r\beta e^{r\beta}B_r} \to z,$$

where z is a fixed positive number. Then

$$\mathbf{P}\{\eta \le z\} \to e^{-z}.$$