On maximum vertex degree in configuration graphs

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We consider a configuration graph where vertex degrees are independent identically distributed random variables. This model is often used for modeling different complex networks such as the Internet, mobile connections, social networks, and others. The random variables $\xi_1, \ldots, \xi_N$ are equal to the degrees of the vertices with numbers $1, \ldots, N$, where N is the number of the graph's vertices. The degrees of vertices are drawn independently from an arbitrary given distribution. Suppose the limit behaviour of this distribution as $k \to \infty$:

$$\mathbf{P}\{\xi_i = k\} \sim \frac{d}{k^g (\ln k)^h},$$

where $i = 1, \ldots, N, \quad d > 0, \quad 2 < g < 3, \quad h \geqslant 0.$

Let

$$\zeta_N = \xi_1 + \ldots + \xi_N \qquad \text{and} \qquad \eta_{(N)} = \max\{\xi_1, \ldots, \xi_N\}.$$

We consider a subset of the graphs under the condition that $\zeta_N = n$. Denote by $\eta_1, \ldots, \eta_N$ the random variables equal to the degrees of vertices in such a conditional random graph. It is evident that these random variables are dependent, and for natural $k_1, \ldots, k_N$ such that $k_1 + \ldots + k_N = n$

$$\mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N | \xi_1 + \ldots + \xi_N = n\}. \quad (1)$$

The equation (1) means that for the random variables $\xi_1, \ldots, \xi_N$ and $\eta_1, \ldots, \eta_N$ the generalized allocation scheme is valid and we can apply the known properties of this scheme to study conditional random graphs.

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We obtained the limit distributions of the maximum vertex degree in these conditional configuration graphs as $N, n \to \infty, (n - MN)/N^{1/(g-1)} \to \infty$, where $M = \mathbf{E}\xi_1$.

Let $B_N = (N(g-1)^h/\ln^h N)^{1/(g-1)}$ and $g(x)$ is the density of the stable law with the exponent $g-1$ and the characteristic function

$$\exp\left\{\frac{\Gamma(3-g)d\cos(\pi(g-1)/2)}{(g-2)(g-1)}|t|^{g-1}\left(1 - i\frac{t}{|t|}\tan(\pi(g-1)/2)\right)\right\},$$

where $\Gamma(x)$ is the Gamma function at the point x. The following theorem holds.

Theorem. *Let $n, N \to \infty, (n-MN)/(N/\ln^h N)^{1/(g-1)} \to \infty$. Then, for any fixed $z > 0$*

$$\mathbf{P}\left\{\frac{n - MN - \eta_{(N)}}{B_N} \leqslant z\right\} \to \int\limits_{-\infty}^{z} g(x)dx.$$