On intermediate value theorem for quantization dimensions

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The quantization of a probability measure is an approximation of a given measure by probability measures with finite supports. The quantization technique finds applications in a number of applied areas. Now the theory of quantization is a substantive section of probability theory (see [1]). The concept of quantization dimension of probability measures is defined within the framework of quantization theory.

Let (X, ρ) be a compact metric space. Denote by $P(X)$ the space of probability measures on X endowed with the Kantorovich–Rubinshtein metric ρ_P defined by the following formula for $\mu, \nu \in P(X)$:

$$\rho_P(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : f \in \mathrm{Lip}_1(X)\},$$

where $\mu(f) = \int f\, d\mu$ and $\mathrm{Lip}_1(X)$ is the set of mappings $f : X \to \mathbb{R}$ such that $|f(x) - f(y)| \le \rho(x, y)$ for all $x, y \in X$. The support $\mathrm{supp}(\mu)$ of a probability measure μ is the least closed subset $A \subset X$ satisfying $\mu(A) = 1$. It is known that for any $n \in \mathbb{N}$ the set $P_n(X) = \{\mu \in P(X) : |\mathrm{supp}(\mu)| \le n\}$ is a closed subset of $P(X)$ and the set

$$P_\infty(X) = \bigcup_{n \in \mathbb{N}} P_n(X)$$

of measures with finite supports is dense in $P(X)$. Therefore, for a given probability measure μ and $\varepsilon > 0$ there exists a measure ν with finite support which is an ε-approximation of μ. We denote by $N(\mu, \varepsilon)$ the least number of points in the support of a such measure ν:

$$N(\mu, \varepsilon) = \min\{n : \rho_P(\mu, P_n(X)) \le \varepsilon\}.$$

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For any probability measure $\mu \notin P_\infty(X)$, the number $N(\mu, \varepsilon)$ increases indefinitely when $\varepsilon \to 0$. The rate of this increase is characterized by the value

$$D(\mu) = \lim_{\varepsilon \to 0} \frac{\log N(\mu, \varepsilon)}{-\log \varepsilon}, \tag{1}$$

which is called the quantization dimension of μ. In fact, $D(\mu)$ is the "switching point" of the limit $\lim_{\varepsilon \to 0} N(\mu, \varepsilon)\varepsilon^\alpha$: for $\alpha < D(\mu)$ this limit equals to ∞, for $\alpha > D(\mu)$ it is equal to 0.

It is known that

$$D(\mu) \leq \dim_B(\mathrm{supp}(\mu)) \tag{2}$$

for any $\mu \in P(X)$, where $\dim_B$ is the box dimension.

Actually, the box dimension $\dim_B F$ is the "quantization dimension" of a closed subset $F \subset X$ with respect to an approximation of F by finite subsets. Let $\exp X$ be the space of nonempty closed subsets of X endowed with the Hausdorf metric ρ_H. As in above, put $\exp_n X = \{G \in \exp X : |G| \leq n\}$ and for each $F \in \exp X$ put $N(F, \varepsilon) = \min\{n : \rho_H(F, \exp_n X) \leq \varepsilon\}$. Then

$$\dim_B F = \lim_{\varepsilon \to 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \tag{3}$$

If the limit (1) (or (3)) does not exist, the upper and lower limits are considered, and we get the upper $\overline{D}(\mu)$ and the lower $\underline{D}(\mu)$ quantization dimensions (or the corresponding box dimensions).

According to inequality (2), the quantization dimensions of probability measures on X are bounded by the box dimension $\dim_B X$. In connection with this inequality, the following question naturally arises: is it true that for any $a \in [0, \dim_B X]$ there exists a probability measure μ_a on X, for which $D(\mu_a) = a$ and $\mathrm{supp}(\mu_a) = X$?

Similar questions about intermediate values are discussed in various branches of mathematics. For example, in topological dimension theory it is known that for any metric compact space X and each natural number $k < \dim X$ there exists a closed subset $F \subset X$ for which $\dim F = k$. But in the non-metrizable case this assertion is false. In measure theory, the following theorem on intermediate values is known: if the measure μ on the set X does not contain atoms, then for any measurable set $A \subset X$ and any $a \in [0, \mu(A)]$ there exists a measurable subset $B \subset A$ whose measure $\mu(B)$ is equal to a.

Our main result it the following theorem which gives a positive solution of the question posed above (for upper quantization dimension):

Theorem. *Let X be a metric compact space. Then for each $a \in [0, \overline{\dim}_B X]$ there exists a measure $\mu_a \in P(X)$ such that $\overline{D}(\mu_a) = a$ and $\mathrm{supp}(\mu_a) = X$.*

It is also proved that on any infinite metric compact space X there exists a probability measure μ with countable support for which $\overline{D}(\mu) = \overline{\dim}_B X$.

For the lower quantization dimension, the question of the validity of similar statements remains open.

References

[1] Graf S., Luschgy H. Foundations of Quantization for Probability Distributions. 2000. Springer-Verlag.