On limit theorems for homogeneous discrete-time nonlinear Markov chains

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Although Markov chains are widespread, and their stability is well studied, sometimes we want to consider the nonlinear models in the sense of dependence on the process distribution. Then, some difficulties with model stability may arise for nonlinear Markov chains. This paper presents some results on the law of large numbers and the central limit theorem for some class of nonlinear Markov chains so that these results may be considered as some stability properties.

Let $(X^\mu_n)_{n\in\mathbb{Z}_+}$ be a nonlinear Markov chain with state space $(E,\mathcal{E})$, $|E|<\infty$, initial distribution $\mathrm{Law}\,(X^\mu_0)=\mu$, $\mu\in\mathcal{P}(E)$ and transition probabilities $P_{\mu_n}(x,B)=\mathbb{P}\left(X^\mu_{n+1}\in B|X^\mu_n=x;\mathcal{L}(X^\mu_n)=\mu_n\right)$, where $x\in E$, $B\in\mathcal{E}$, $n\in\mathbb{Z}_+$ and $\mu_n:=\mathrm{Law}(X^\mu_n)$. Therefore, the transition kernel depends on both the state of the process at the moment n and the distribution of the process at that moment.

First of all, the existence and uniqueness of an invariant measure for nonlinear Markov chains is not as simple as in case of ordinary Markov chains. There are some results that prove the fact that such processes do have a unique invariant measure if they satisfy the following conditions:

$$\sup_{\mu,\nu\in\mathcal{P}(E)}\|P_\mu(x,\cdot)-P_\nu(y,\cdot)\|_{TV}\le 2(1-\alpha),\quad 0<\alpha<1,\quad x,y\in E,\tag{1}$$

$$\|P_\mu(x,\cdot)-P_\nu(x,\cdot)\|_{TV}\le\lambda\|\mu-\nu\|_{TV},\quad \lambda\in[0,\alpha],\quad x\in E,\quad \mu,\nu\in\mathcal{P}(E).\tag{2}$$

Considering such class of the processes, that satisfy these conditions, we can also establish the law of large numbers and central limit theorem.

Theorem 1. *Let X^μ_k be a nonlinear Markov chain defined on a measurable finite state space $(E,\mathcal{E})$ that satisfies the conditions (1), (2) and $g\in\mathcal{C}_b$.*

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Denote $S_n = \sum_{k=0}^{n-1} g(X_k^\mu)$*, then the sequence* S_n *satisfies the law of large numbers, such as,* $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mathbb{E}[g(X_0^\pi)],$$

where $\mathbb{E}[g(X_n^\pi)] = \int g(x)\pi(dx)$*,* X_k^π *is a copy of* X_k^μ *with initial distribution equal to the invariant measure* π*.*

Theorem 2. *Let* X_k^μ *be a nonlinear Markov chain defined on a measurable finite state space* $(E, \mathcal{E})$ *that satisfies the conditions* (1)*,* (2) *and* $g \in \mathcal{C}_b$*. Denote* X_k^π *a copy of* X_k^μ *with initial distribution equal to the invariant measure* π*,*

$$S_n = \sum_{i=0}^{n-1} \left(f(X_i^\mu) - \mathbb{E}[f(X_0^\pi)]\right),$$

$$\sigma^2 = \mathbb{E}_\pi \left[\sum_{0 \le i \le j \le \infty} \left(f(X_i^\mu) - \mathbb{E}[f(X_0^\pi)]\right) \left(f(X_j^\mu) - \mathbb{E}[f(X_0^\pi)]\right) \right].$$

Then, the nonlinear Markov chain satisfies the central limit theorem,

$$\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}} \right) \right] \to \mathbb{E}\left[g(\eta)\right], \quad \eta \sim \mathcal{N}(0, \sigma^2).$$